

BRST Cohomology and Its Application to QED

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Abstract

We construct the BRST cohomology under a positive-definite inner product and obtain the Hodge decomposition theorem at a non-degenerate state vector space V . The harmonic states isomorphic with a BRST cohomology class correspond to the physical Hilbert space with positive norm as long as the completeness of Q_{BRST} is satisfied. We explicitly define a “co-BRST” operator and analyze the quartet mechanism in QED.

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I. INTRODUCTION

The covariant quantization of constrained systems and the renormalization of gauge theories heavily depend on the BRST approach [1,2]. This approach extends the phase space, including the anticommuting ghost variables. Thus, we must project out all the physical states in a positive-definite Hilbert space in order to recover the probabilistic interpretation of the quantum theory. We achieve this goal by asking for the BRST invariant states of ghost number zero [3]. However, the nilpotency of the BRST operator gives the equivalence classes of physical states, which naturally leads us into the cohomology group interpretation about the physical Hilbert space. This use of the BRST cohomology to characterize the physical states can be strengthened more by introducing the adjoint operation of the BRST operator [4–6]. This statement is analogous to the Hodge decomposition theorem of differential geometry, which naturally leads to an isomorphism between the space of the cohomology group and the space of harmonic forms [7]. The physical Hilbert space can be identified with the harmonic states which have positive-definite norms as long as the disastrous paired singlet with non-zero ghost number is absent. Plausible arguments exist for the absence of singlet pairs in the actual models of gauge theories [3]. Following Refs. 4 and 5, we will show the apparent correspondence between the irreducible representation of the BRST algebra and the BRST cohomology.

In Sec. II, we construct the BRST cohomology under a positive-definite inner product defined on a non-degenerate state space graded with a ghost number and obtain the Hodge decomposition theorem. In order to define a positive-definite inner product, we introduce the metric on the state space V , which is the analogue of the Euclidean complex conjugation \mathbf{C} in Ref. 5 and the time reversal operation T in Ref. 8. This metric matrix provides us an isomorphism between the subspaces V_p and V_{-p} of the respective ghost numbers $+p$ and $-p$, and this isomorphism induces an isomorphism in the BRST cohomology. The metric matrix η plays an important role in the proof of the quartet mechanism and the “split a pair” mechanism. From the metric matrix, we easily find that $Tr \eta$ is equal to the dimension of

the BRST singlet subspace V_S with a positive-definite norm. The new positive definite inner product provides us the orthogonal decomposition of the state space V . This decomposition into a sum of linearly independent subspaces is just the Hodge decomposition theorem. We will show how the illuminating example in Ref. 5 can be realized through the quartet mechanism.

In Sec. III, we explicitly analyze the BRST cohomology of QED using the mode expansion of the field operators. We find the “co-BRST” operator defined in Sec. II and show the quartet mechanism and the pair-splitting mechanism taking an analogy with Ref. 5.

II. BRST COHOMOLOGY

Let us assume that a state vector space V with an indefinite metric is non-degenerate and equipped with complete basis vectors $\{|h_j\rangle\}$. Then, an arbitrary vector $|x\rangle$ in V can be represented uniquely as

$$|x\rangle = \sum_j x_j |h_j\rangle. \quad (2.1)$$

The metric matrix η on the vector space V is defined by

$$\eta_{ij} = \langle h_i | h_j \rangle, \quad (2.2)$$

which is Hermitian,

$$\eta^\dagger = \eta, \quad (2.3)$$

and the non-degeneracy of the space V can be expressed as

$$\det \eta \neq 0. \quad (2.4)$$

The representation t on V of a linear operator T is defined by

$$T|h_j\rangle = \sum_k |h_k\rangle t_{kj}. \quad (2.5)$$

The matrix representations t and \bar{t} of T and T^\dagger , respectively, should satisfy the condition [4]

$$\eta \bar{t} = t^\dagger \eta. \quad (2.6)$$

Here, t^\dagger is the Hermitian conjugate of the matrix t . In particular, when the operator T is Hermitian, that is $T^\dagger = T$, we have

$$\eta t = t^\dagger \eta \quad (2.7)$$

so that the matrix t is not necessarily Hermitian. On the state vector space V , we can introduce the basis transformation U by

$$|h'_j\rangle = \sum_k |h_k\rangle u_{kj}. \quad (2.8)$$

Under the transformation in Eq. (2.8), the representation matrix t and the metric matrix η transform as follows:

$$t \longrightarrow t' = u^{-1} t u, \quad \eta \longrightarrow \eta' = u^\dagger \eta u. \quad (2.9)$$

According to the above statement, we now introduce the important representation matrices q and n of the Hermitian BRST operator Q and the anti-Hermitian ghost number operator N_{gh} , respectively. The Hermiticity conditions for q and n assume the following forms:

$$\eta q = q^\dagger \eta, \quad (2.10)$$

$$\eta n = -n^\dagger \eta. \quad (2.11)$$

We shall choose a representation in which n is diagonalized with integral elements. Note that it is always possible to find an appropriate u in Eq. (2.8) which brings η into the standard form [4,9]:

$$\eta^2 = 1. \quad (2.12)$$

Equation (2.11) also shows that the metric matrix η defines an isomorphism between the subspaces V_p and V_{-p} of respective ghost numbers $+p$ and $-p$. In the BRST cohomology, a

natural way to use the Hodge theory argument is to introduce a new positive-definite inner product [8] defined by

$$(x|y) \equiv \langle x|\eta y \rangle \quad \text{for } |x\rangle, |y\rangle \in V. \quad (2.13)$$

Note that the new inner product, Eq. (2.13), is non-degenerate by Eq. (2.4) and that the norms of a vector $|x\rangle$ in the state space V with respect to the new inner product $(| \rangle)$ and the physical inner product $\langle | \rangle$ can be expressed as

$$(x|x) = \sum |x_i|^2, \quad (2.14)$$

$$\langle x|x \rangle = \sum \eta_{ij} x_i^* x_j. \quad (2.15)$$

Thus, the adjoint operator \tilde{T}^\dagger of T in the new metric defined by $(x|Ty) = (\tilde{T}^\dagger x|y)$ satisfies $\tilde{T}^\dagger \equiv \eta T^\dagger \eta$, where $\langle x|Ty \rangle = \langle T^\dagger x|y \rangle$. Note that the matrix representation \tilde{t}^\dagger of \tilde{T}^\dagger is equal to the matrix t^\dagger . We will replace the notation \tilde{T}^\dagger by T^\dagger , since we shall treat only the (anti-)Hermitian operators, so that it raises no confusion.

It will be more obvious later that the metric matrix η has the same roles and philosophy as the Euclidean complex conjugation \mathbf{C} in Ref. 5 and the time reversal operation T in Ref. 8. Since the BRST operator Q cannot be self-adjoint with respect to the inner product in Eq. (2.13), it is convenient to introduce the adjoint operator Q^\dagger of Q called the “co-BRST” operator defined by

$$(Qx|y) = (x|Q^\dagger y); \quad (2.16)$$

its matrix representation is given by Eq. (2.10):

$$q^\dagger = \eta q \eta. \quad (2.17)$$

Let us introduce the “Laplacian” operator Δ defined by

$$\Delta = \{Q, Q^\dagger\}. \quad (2.18)$$

Then one finds that Q , Q^\dagger , and Δ satisfy the supersymmetrylike algebra

$$\{Q, Q^\dagger\} = \Delta, \quad [\Delta, Q] = 0, \quad [\Delta, Q^\dagger] = 0. \quad (2.19)$$

Now the BRST cohomology algebra is given by

$$\begin{aligned} [n, q] &= q, & [n, q^\dagger] &= -q^\dagger, \\ q^2 &= q^{\dagger 2} = 0, & \{q, q^\dagger\} &= \delta, \end{aligned} \quad (2.20)$$

where δ is a matrix representation of the operator Δ and the operators shall be represented in the basis in which n is diagonalized with integer elements. Now, we introduce two subspaces of V by

$$Im\ Q \equiv QV = \{|g\rangle \equiv q|x\rangle \mid |x\rangle \in V\}, \quad (2.21)$$

$$Im\ Q^\dagger \equiv Q^\dagger V = \{|f\rangle \equiv q^\dagger|x\rangle \mid |x\rangle \in V\}. \quad (2.22)$$

Due to the nilpotency of q and q^\dagger , all the states in the BRST doublet space have zero norms with respect to the physical metric, and the following properties are satisfied:

$$q|g\rangle = 0, \quad \text{for } |g\rangle \in Im\ Q, \quad (2.23)$$

$$q^\dagger|f\rangle = 0, \quad \text{for } |f\rangle \in Im\ Q^\dagger. \quad (2.24)$$

From the positive-definite inner product in Eq. (2.13),

$$(g|g) = \sum |g_i|^2 \neq 0 \quad \text{and} \quad (f|f) = \sum |f_i|^2 \neq 0, \quad (2.25)$$

and Eq. (2.25) implies that $Q(\eta|g\rangle) = q\eta|g\rangle \neq 0$ and $Q^\dagger(\eta|f\rangle) = q^\dagger\eta|f\rangle \neq 0$. These mean that

$$\eta\ Im\ Q = Im\ Q^\dagger \quad \text{and} \quad \eta\ Im\ Q^\dagger = Im\ Q \quad (2.26)$$

since the metric matrix η is non-singular. Note that the above metric conjugate pairs must have opposite ghost numbers because of Eq. (2.11). On the other hand, $(f|g) = (g|f) = 0$.

Next, we define the singlet space generated by the “harmonic” state defined by $Ker\ \Delta$. This singlet space V_S is equivalent to the statement

$$V_S = \{|x\rangle \mid q|x\rangle = q^\dagger|x\rangle = 0, \quad |x\rangle \in V\} \quad (2.27)$$

due to the positive definiteness of the new inner product, Eq. (2.13). Then, V_S is orthogonal to $Im Q$ and $Im Q^\dagger$ and satisfies

$$\eta V_S = V_S. \quad (2.28)$$

Of course, the metric conjugate pairs in the singlet states also have opposite ghost numbers. Therefore, the new positive-definite inner product in Eq. (2.13) provides us the orthogonal decomposition about the state space V , which is a sum of linearly independent subspaces as follows [4–6]:

$$V = V_S \oplus V_D = Ker \Delta \oplus Im Q \oplus Im Q^\dagger. \quad (2.29)$$

This decomposition on the vector space V is just the Hodge decomposition theorem. According to Eq. (2.29), the BRST doublet space V_D satisfies the following properties:

$$Im Q = q Im Q^\dagger \quad \text{and} \quad Im Q^\dagger = q^\dagger Im Q. \quad (2.30)$$

According to Eqs. (2.29) and (2.30), the condition of the physical subspace V_{phys} defined by $Ker Q$ [3] is equivalent to

$$V_{phys} = Ker \Delta \oplus Im Q. \quad (2.31)$$

Thus, the BRST doublet pairs are split in the physical subspace V_{phys} . If the subspace $Ker \Delta$ has a positive norm, the physical space V_{phys} cannot contain a negative norm state because of the divorce of metric conjugate pairs, Eq. (2.26), so that a state in V_{phys} is a positive norm or a zero norm.

We define the p -th BRST cohomology group in subspace V_p by the BRST equivalence class of ghost number p , that is, the kernel of Q modulo its image:

$$\begin{aligned} H^p(V) &\equiv Ker^p Q / Im^p Q, \\ &\cong Ker^p \Delta, \quad p = 0, \dots, \pm \dim G, \end{aligned} \quad (2.32)$$

where G is the structure group of the gauge symmetry under consideration. The Hodge decomposition theorem directly leads to the isomorphism between the p -th BRST cohomology

space $H^p(V)$ and the harmonic state space $Ker^p \Delta$. Since the metric matrix η defines the isomorphism between the subspaces V_p and V_{-p} , the metric η also induces an isomorphism in BRST cohomology, namely,

$$H^p(V) \cong H^{-p}(V). \quad (2.33)$$

With these preliminaries we will study the irreducible representation of BRST cohomology algebra and the condition of the positivity of the physical state space in BRST quantization.

1. BRST Doublet Representation

The basis of the BRST doublet representation consists of metric conjugate pairs in the BRST doublet space, Eq. (2.30), labeled by the ghost number:

$$\begin{aligned} (N+1, Im Q) &\equiv (1, 0, 0, 0), \quad (N, Im Q^\dagger) \equiv (0, 1, 0, 0), \\ (-N, Im Q) &\equiv (0, 0, 1, 0), \quad (-N-1, Im Q^\dagger) \equiv (0, 0, 0, 1). \end{aligned} \quad (2.34)$$

The irreducible representation of the BRST cohomology algebra Eq. (2.20) is given by

$$\begin{aligned} \eta &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad n = \begin{pmatrix} N+1 & 0 & 0 & 0 \\ 0 & N & 0 & 0 \\ 0 & 0 & -N & 0 \\ 0 & 0 & 0 & -N-1 \end{pmatrix}, \\ q &= a \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad q^\dagger = a \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \delta = a^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \quad (2.35)$$

where a is a coefficient to be determined. The matrix representation δ of the Laplacian operator Δ in Eq. (2.35) shows us that no quartet member appears in the BRST cohomology space: *Quartets are always confined!*

2. BRST Singlet Representation

There are two kinds of singlet representations in the space, Eq. (2.27), with the ghost numbers $N = 0$ and $N \neq 0$. For convenience, we consider the two cases together:

$$(0, Ker \Delta), (N, Ker \Delta), (-N, Ker \Delta). \quad (2.36)$$

The irreducible representation of the BRST cohomology algebra, Eq. (2.20), is given by

$$\begin{aligned} \eta &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & n &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & -N \end{pmatrix}, \\ q &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & q^\dagger &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \delta &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.37)$$

From the Eqs. (2.35) and (2.37), we see that the dimension of the harmonic state space $Ker \Delta$ is not less than $Tr \eta$ and that the dimension of the space $V_S^{(+)}$ generated by the basis $(0, Ker \Delta)$ is equal to the index $Tr \eta$. If Q is complete, that is, the set of all the zero-norm states in $Ker Q$ is $Im Q$ [5], the completeness condition of Q is obviously equivalent to the condition of the absence of the paired singlet with $N \neq 0$. If this condition is satisfied, the norms of all the states in $Ker Q$ must have the same sign, as shown in Ref. 5. Therefore, we can always choose the sign of $Ker Q$ as positive. Then $Tr \eta$ is equal to the dimension of the positive-definite Hilbert space as claimed in Ref. 5. In addition, the “split a pair” mechanism is complete in V_{phys} so that negative norm states cannot be generated in V_{phys} . Consequently, the BRST cohomology space is 0-norm-state free, thus proving the “no-ghost” theorem. We, thus, conclude that the cohomology space $H(V)$ isomorphic with the harmonic space $Ker \Delta$ has a positive-definite metric so that the physical Hilbert space has a positive-definite norm as long as the paired singlet is absent.

In order to show how the illustrative example in Ref. 5 can be realized through the quartet mechanism in the BRST doublet representation, Eq. (2.35), it will be interesting to take the basis transformation, Eq. (2.8), into the basis for the metric matrix η to be

diagonalized. Then, the basis of the BRST doublet representation consists of the metric eigenstates

$$\begin{aligned}\vec{k} &\equiv \frac{1}{2}(1, -1, 1, -1), \quad \vec{l}' \equiv \frac{1}{2}(-1, -1, 1, 1), \\ \vec{l} &\equiv \frac{1}{2}(1, 1, 1, 1), \quad \vec{k}' \equiv \frac{1}{2}(-1, 1, 1, -1).\end{aligned}\tag{2.38}$$

The irreducible representation of the BRST cohomology algebra is given by

$$\begin{aligned}\eta &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad n = \begin{pmatrix} 0 & 0 & \frac{1}{2} & -N - \frac{1}{2} \\ 0 & 0 & -N - \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -N - \frac{1}{2} & 0 & 0 \\ -N - \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}, \\ q &= \frac{a}{2} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad q^\dagger = \frac{a}{2} \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}, \quad \delta = a^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.\end{aligned}\tag{2.39}$$

Then, analogies with the example in Ref. 5 can be definitely realized through the quartets in Eq. (2.39) in the BRST cohomology. Of course, the same words can also be applied to the representation in Eq. (2.35). First, notice the following facts:

$$\begin{aligned}n\vec{k} &= (N+1)\vec{k}, \quad n\vec{k}' = -(N+1)\vec{k}', \\ n\vec{l} &= -N\vec{l}, \quad n\vec{l}' = N\vec{l}',\end{aligned}\tag{2.40}$$

$$\eta\vec{k} = \vec{k}', \quad \eta\vec{l} = \vec{l}'.\tag{2.41}$$

Second, the matrices q and q^\dagger can be written as

$$q = a(\vec{k}\vec{l} + \vec{l}\vec{k}), \quad q^\dagger = \eta q \eta = a(\vec{k}'\vec{l}' + \vec{l}'\vec{k}')\tag{2.42}$$

where $\vec{k}\vec{l} \equiv k^i l_j$. Then, we easily find that

$$\begin{aligned}Ker Q &= \{\vec{k} = \frac{1}{a}q\vec{l}', \quad \vec{l} = \frac{1}{a}q\vec{k}'\} = Im Q, \\ Ker Q^\dagger &= \{\vec{k}' = \frac{1}{a}q^\dagger\vec{l}, \quad \vec{l}' = \frac{1}{a}q^\dagger\vec{k}\} = Im Q^\dagger.\end{aligned}\tag{2.43}$$

Therefore, if the singlet pairs with non-zero ghost number are absent, the completeness of Q and the “split a pair” mechanism are also obvious in the representation of Eq. (2.39). Slight differences with the Ref. 5 exist in the above identifications about quartet states in that our members are more appropriate since q increases the ghost number of a state by one unit while η connects states with opposite ghost numbers. In the next section, we will study the explicit example of the BRST cohomology discussed in Sec. II in the context of QED.

III. BRST COHOMOLOGY IN QED

Consider the (anti-)BRST invariant effective QED Lagrangian.

$$\begin{aligned}\mathcal{L}_{eff} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2}\bar{s}s(A_\mu^2 + \alpha\bar{c}c) \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + A_\mu\partial^\mu b + \frac{\alpha}{2}b^2 - \partial_\mu\bar{c}\partial^\mu c\end{aligned}\quad (3.1)$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative with the metric $g_{\mu\nu} = (1, -1, -1, -1)$. (Our BRST treatments are parallel with those of Baulieu’s paper [2].) This effective Lagrangian has the rigid symmetry under the following BRST transformation:

$$\begin{aligned}sA_\mu &= \partial_\mu c, \quad sc = 0, \\ s\bar{c} &= b, \quad sb = 0, \\ s\psi &= -iec\psi.\end{aligned}\quad (3.2)$$

We introduced an auxiliary field b to achieve off-shell nilpotency of the BRST transformation. Then, the nilpotent conserved Nöther charges generated by the BRST transformation in Eq. (3.2) read as

$$Q = \int d^3x \{-(\partial_i F^{i0} - \rho)c - b\dot{c}\} \quad (3.3)$$

where ρ is a charge density defined by

$$\rho = e\bar{\psi}\gamma_0\psi. \quad (3.4)$$

The transformations in Eq. (3.2) now can be defined as follows: $s\mathcal{F}(x) = i[Q, \mathcal{F}(x)]$ where the symbol $[,]$ is the graded commutator. In the language of quantum field theory, the BRST operator Q is the generator of the quantum gauge transformation.

For simplicity, we only consider the free Maxwell theory since the matter fields are not essential in the BRST cohomology. Using the mode expansion of field operators [1], we find the following expression about the BRST operator Q :

$$Q = \sum_{\mathbf{k}} (a_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} + a_{\mathbf{k}}^{\dagger} c_{\mathbf{k}}) \quad (3.5)$$

where $a_{\mathbf{k}}(a_{\mathbf{k}}^{\dagger})$ is a linear combination of the longitudinal photon $a_3(a_3^{\dagger})$ and the temporal photon $a_0(a_0^{\dagger})$ of momentum \mathbf{k} and is given by $a_{\mathbf{k}} = a_{3\mathbf{k}} - a_{0\mathbf{k}}$ ($a_{\mathbf{k}}^{\dagger} = a_{3\mathbf{k}}^{\dagger} - a_{0\mathbf{k}}^{\dagger}$) with its conjugate variable $b_{\mathbf{k}}^{\dagger} \equiv \frac{1}{2}(a_{3\mathbf{k}}^{\dagger} + a_{0\mathbf{k}}^{\dagger})$ ($b_{\mathbf{k}} \equiv \frac{1}{2}(a_{3\mathbf{k}} + a_{0\mathbf{k}})$). Notice canonical quantization leads to

$$\begin{aligned} [a_{\mathbf{k}}, b_{\mathbf{k}'}^{\dagger}] &= \delta_{\mathbf{k}\mathbf{k}'}, \quad [b_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger}] = \delta_{\mathbf{k}\mathbf{k}'}, \\ \{c_{\mathbf{k}}, \bar{c}_{\mathbf{k}'}^{\dagger}\} &= \delta_{\mathbf{k}\mathbf{k}'}, \quad \{\bar{c}_{\mathbf{k}}, c_{\mathbf{k}'}^{\dagger}\} = \delta_{\mathbf{k}\mathbf{k}'}; \end{aligned} \quad (3.6)$$

the other (anti-)commutators vanish.

Now, we will apply the BRST cohomology described in Sec. II to QED. We define the physical vacuum $|0\rangle$ as the state annihilated by all the destruction operators:

$$a_{\mathbf{k}}|0\rangle = b_{\mathbf{k}}|0\rangle = c_{\mathbf{k}}|0\rangle = \bar{c}_{\mathbf{k}}|0\rangle = 0. \quad (3.7)$$

We construct the Fock space Ω which has an indefinite metric and is decomposed into the tensor product

$$\Omega = \Omega_T \otimes \Omega_F.$$

Here, Ω_T is the space $V_S^{(+)}$ discussed in Sec. II where, for example, a transverse photon lives, and the subspace Ω_F of the unphysical sectors is generated by

$$|m, n; k, l\rangle \equiv \frac{1}{\sqrt{m!n!}} a^{\dagger m} b^{\dagger n} c^{\dagger k} \bar{c}^{\dagger l} |0\rangle. \quad (3.8)$$

Since the conjugate pairs of the (anti-)commutators in Eq. (3.6) will be adjoints of one another under the inner product in Eq. (2.13), one can easily find the following relations:

$$\begin{aligned}\eta a_{\mathbf{k}} \eta &= b_{\mathbf{k}}, & \eta a_{\mathbf{k}}^{\dagger} \eta &= b_{\mathbf{k}}^{\dagger}, \\ \eta c_{\mathbf{k}} \eta &= \bar{c}_{\mathbf{k}}, & \eta c_{\mathbf{k}}^{\dagger} \eta &= \bar{c}_{\mathbf{k}}^{\dagger}.\end{aligned}\tag{3.9}$$

Then, the explicit form of the adjoint operator Q^{\dagger} of Q is

$$Q^{\dagger} = \sum_{\mathbf{k}} (b_{\mathbf{k}} \bar{c}_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}^{\dagger} \bar{c}_{\mathbf{k}}).\tag{3.10}$$

This operator Q^{\dagger} is consistent with the definition in Eq. (2.16) for the “co-BRST” operator. The “Laplacian” operator Δ defined in Sec. II can then be described by

$$\Delta = \sum_{\mathbf{k}} (a_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \bar{c}_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} + c_{\mathbf{k}}^{\dagger} \bar{c}_{\mathbf{k}}),\tag{3.11}$$

and this is the number operator N_{unphys} for the unphysical modes. Thus, the operator Δ is essentially the part of the Hamiltonian with the unphysical fields if it is multiplied by the frequency $\omega_{\mathbf{k}}$ for each mode of momentum \mathbf{k} . This property of our BRST cohomology is reminiscent of the definition in Ref. 8 about a quantum cohomology whose cohomology classes correspond to quantum ground states. From Eq. (3.11), we can find the coefficient a in Eq. (2.35) or Eq. (2.39); this is just the number of unphysical particles. The quartet members are characterized by the eigenvalues of the number operator N_{unphys} .

Let us introduce the “total” ghost number operator N_{gh}^{total} defined by

$$\begin{aligned}N_{gh}^{total} &= N_{gh}^{(0)} + N_{gh}^{(1)}, \\ N_{gh}^{(0)} &= \sum_{\mathbf{k}} (a_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - b_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}), & N_{gh}^{(1)} &= \sum_{\mathbf{k}} (\bar{c}_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} - c_{\mathbf{k}}^{\dagger} \bar{c}_{\mathbf{k}})\end{aligned}\tag{3.12}$$

where $N_{gh}^{(1)}$ is just the ordinary ghost number operator N_{gh} introduced in Sec. II. The states $|m, n; k, l\rangle$ in Eq. (3.8) are eigenstates of $N_{gh}^{(0)}$, $N_{gh}^{(1)}$ and Δ with eigenvalues of $m - n$, $k - l$, and $m + n + k + l$, respectively. From Eq. (3.12) and Eq. (3.6), one can easily find the following properties:

$$\begin{aligned}
[N_{gh}^{(0)}, Q] &= Q, \quad [N_{gh}^{(1)}, Q] = Q, \\
[N_{gh}^{(0)}, Q^\dagger] &= -Q^\dagger, \quad [N_{gh}^{(1)}, Q^\dagger] = -Q^\dagger, \\
[N_{gh}^{total}, Q] &= 2Q, \quad [N_{gh}^{total}, Q^\dagger] = -2Q^\dagger, \quad [N_{gh}^{total}, \Delta] = 0.
\end{aligned} \tag{3.13}$$

The bases corresponding to the BRST doublet representation in Eq.(2.34) consist of the following Fock states (using the same notations defined by Eqs. (2.34) and (2.38)):

$$\begin{aligned}
(I) : \quad |0, m; 1, 0 \rangle &= (1, 0, 0, 0), \quad |0, m+1; 0, 0 \rangle = (0, 1, 0, 0), \\
|m+1, 0; 0, 0 \rangle &= (0, 0, 1, 0), \quad |m, 0; 0, 1 \rangle = (0, 0, 0, 1),
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
(II) : \quad |n+1, m; 1, 0 \rangle &= (1, 0, 0, 0), \\
\sqrt{\frac{m+1}{n+m+2}} |n+1, m+1; 0, 0 \rangle &- \sqrt{\frac{n+1}{n+m+2}} |n, m; 1, 1 \rangle = (0, 1, 0, 0), \\
\sqrt{\frac{m+1}{n+m+2}} |m+1, n+1; 0, 0 \rangle &+ \sqrt{\frac{n+1}{n+m+2}} |m, n; 1, 1 \rangle = (0, 0, 1, 0), \\
|m, n+1; 0, 1 \rangle &= (0, 0, 0, 1).
\end{aligned} \tag{3.15}$$

The bases (I) in Eq. (3.14) and (II) in Eq. (3.15), respectively, correspond to the eigenvalues $a^2 = m+1$ and $a^2 = n+m+2$ of N_{unphys} or Δ in the representation of Eq. (2.35). They exhaust all zero-norm states in $Ker Q$ and form the quartets in the $(m+1)$ - and $(n+m+2)$ -unphysical sectors:

$$\begin{aligned}
\eta &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad n_I^{(0)} = \begin{pmatrix} -m & 0 & 0 & 0 \\ 0 & -m-1 & 0 & 0 \\ 0 & 0 & m+1 & 0 \\ 0 & 0 & 0 & m \end{pmatrix}, \\
q &= \begin{pmatrix} \sqrt{m+1} \\ \sqrt{n+m+2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad q^\dagger = \begin{pmatrix} \sqrt{m+1} \\ \sqrt{n+m+2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (3.16) \\
\delta &= \begin{pmatrix} m+1 \\ n+m+2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

$n_I^{(0)}$ in Eq. (3.16) is the matrix representation for states (I) of the operator $N_{gh}^{(0)}$ in Eq. (3.12), and the representation for states (II) can be obtained by the replacement $m \longrightarrow m - n - 1$. The matrix structures of the representation in Eq. (3.16) clearly show consistent results, such as the isomorphism between V_p and V_{-p} and the “split a pair” and quartet mechanisms discussed in Sec. II. Therefore, these quartet members disappear in the BRST cohomology. No paired singlet exists and so the BRST singlet states come only from the harmonic states in $(0, Ker\Delta)$. Consequently, the physical Hilbert space V_S is expressed by the Fock space $\Omega = \Omega_T \otimes |0\rangle_F$ which has a positive-definite norm. In the metric-diagonal basis of Eq. (2.38), we also ensure these conclusions along the equivalent logics, Eqs. (2.40)-(2.43).

IV. DISCUSSION

We have shown that there is a straightforward way to isolate the physical Hilbert space with a positive-definite metric through the BRST cohomology. Using the co-BRST operator, we have obtained the Hodge decomposition theorem under a positive-definite inner product and have uniquely chosen the harmonic representative with a positive-definite norm as long

as Q_{BRST} is complete.

In general, the BRST cohomology algebra cannot prevent the invasion of a paired singlet with non-zero ghost number, and the “physical” meaning of the higher cohomology groups $H^p(V)$, $p \neq 0$, is not obvious. If the nontrivial higher cohomologies appeared in the theory, it would be a very interesting problem to give their “physical” interpretations.

Our BRST cohomology is quite different from the cohomology in the recent literature [10,11], which cannot be applied to the problem to directly isolate the physical state with positive-definite norm. Our definition about a “co-BRST” operator is quite similar to the “dual BRST” operator in Ref. 12.

Our “co-BRST” operator defined in QED is not Lorentz invariant because we have flipped the sign of the time-directional operators, but it is a conserved operator that commutes with Hamiltonian, so our recipes for characterizing the physical Hilbert space using the Hodge decomposition theorem have physical significance. An explicit construction of the BRST cohomology in Fock space when the (self-)interaction is present is, in general, a very difficult problem and we have no solutions to this problem. However, see, for example, Ref. 13 which analyzed the unitarity of the S-matrix and the positivity of the physical states norm in the subspace of asymptotic states under the assumption about the asymptotic completeness. Reference 14 uses a similar structure to that in our approach to construct the state space in the BRST quantization, but the state space is not completely characterized because they did not use the some explicit form of the metric as ours.

We also found [15] the local, but non-covariant, symmetries in Abelian gauge theories. The BRST-like charge Q^\perp constructed in Ref. 11 corresponds to the “co-BRST” operator in Ref. 10, and there is no reason to abandon locality, unlike the claim of the Ref. 11.

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